SOME IDENTITIES OF POLYNOMIALS ARISING FROM UMBRAL CALCULUS

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Abstract. In this paper, we study some properties of associated sequences in umbral calculus. From these properties, we derive new and interesting identities of several kinds of polynomials.

1. Introduction

We recall that the Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt}=e^{B(x)t}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!},\quad (see[7,8]),$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$.
In the special case, $x=0$, $B_n(0)=B_n$ are called the n-th Bernoulli numbers $(see[1-14])$.
For $r\in\mathbb{Z}_+$, the higher order Bernoulli polynomials are also defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\underbrace{\left(\frac{t}{e^t-1}\right)\cdots\left(\frac{t}{e^t-1}\right)}_{r-times}e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!}.$$

In the special case, $x=0$, $B_n^{(r)}(0)=B_n^{(r)}$ are called the n-th Bernoulli numbers of order r $(see\,[5,6])$. From the definition of Bernoulli numbers, we note that

$$B_0=1,\quad (B+1)^n-B_n=\delta_{1,n},\quad (see[7,8,10]),$$

where $\delta_{n,k}$ is the Kronecker's symbol.
As is well Known, the Euler and higher-order Euler polynomials are also defined by the generating functions as follows:

$$\frac{2}{e^t+1}e^{xt}=e^{E(x)t}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},$$

and

$$\left(\frac{2}{e^t+1}\right)^r e^{xt}=\underbrace{\left(\frac{2}{e^t+1}\right)\cdots\left(\frac{2}{e^t+1}\right)}_{r-times}e^{xt}=\sum_{n=0}^{\infty}E_n^{(r)}(x)\frac{t^n}{n!},$$

with the usual convention about replacing $\left(E^{(r)}(x)\right)^n$ by $E_n^{(r)}(x)$ $(see[3,4,5,6])$. Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F}=\{f(t)=\sum_{n=0}^{\infty}\frac{a_k}{k!}t^k \mid a_k\in\mathbb{C}\}.$$

Let $\mathbb{P}=\mathbb{C}[t]$ and let $\mathbb{P}^*$ be the vector space of all linear functional on $\mathbb{P}$. Now, we use the notation $< L \mid p(x) >$ to denote the action of a linear functional L on a polynomial $p(x)$ $(see[4,11])$.
The formal power series

$$f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\in\mathcal{F},\quad (see[4,11]),$$

define a linear functional on $\mathbb{P}$ by setting

$$< f(t) \mid x^n >= a_n,\quad for\quad all\quad n\geq 0,\quad (see[4,11]).$$

Thus, we have

$$< t^k \mid x^n >= n!\delta_{n,k},\quad (see[4]).$$

Let $f_L(t)=\sum_{k=0}^{\infty}\frac{<L \mid x^k>}{k!}t^k$. Then, we note that $< f_L(t) \mid x^n >=< L \mid x^n >$ and so as linear functionals $L=f_L(t)$. It is known in [11] that the map $L\mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ on to $\mathcal{F}$. Henceforth, $\mathcal{F}$ will denote both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$ and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We shall call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra and modern classical umbral calculus can be described as a systematic study of the class of Sheffer sequences $(see[11])$. The order $O(f(t))$ of the power series $f(t)\neq 0$ is the smallest integer k for which a_k does not vanish. The series $f(t)$ has a multiplicative inverse, denoted by $f(t)^{-1}$ or $\frac{1}{f(t)}$ if and only if $O(f(t))=0$. Such a series is called invertible series. A series $f(t)$ for which $O(f(t))=1$ is called a delta series $(see[4,11])$. Let $f(t),g(t)\in\mathcal{F}$. Then, we see that

$$< f(t)g(t) \mid p(x) >=< f(t) \mid g(t)p(x) >=< g(t) \mid f(t)p(x) >,\quad (see[11]).$$

In [11], we note that for all $f(t)$ in $\mathcal{F}$

$$f(t)=\sum_{k=0}^{\infty}\frac{< f(t) \mid x^k >}{k!}t^k,$$

and for all polynomials $p(x)$

$$p(x)=\sum_{k=0}^{\infty}\frac{< t^k \mid p(x) >}{k!}x^k.$$

Thus, we get

$$p^{(k)}(x) = \frac{d^k p(x)}{dx^k} = \sum_{l=k}^{\infty} \frac{1}{(l-k)!} < t^l \mid p(x) > x^{l-k},$$

and

$$p^{(k)}(0) =< t^k \mid p(x) > \quad and < 1 \mid p^{(k)}(x) >= p^{(k)}(0).$$

From this, we have $t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, (k \geq 0)$. It is not difficult to show that $e^{yt}p(x) = p(x+y)$ $(see[4, 11])$. Let $S_n(x)$ be a polynomial with deg $S_n(x) = n$, $f(t)$ a delta series and let $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$ of polynomials with $< g(t)f(t)^k \mid S_n(x) >= n!\delta_{n,k}, (n, k \geq 0)$. The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. If $S_n(x) \sim (1, f(t))$, then $S_n(x)$ is called the associated sequences for $f(t)$, or $S_n(x)$ is associated to $f(t)$. If $S_n(x) \sim (g(t), t)$, then $S_n(x)$ is called the Appell sequence for $g(t)$ or $S_n(x)$ is Applell for $g(t)$ $(see[4, 11])$. For $p(x) \in \mathbb{P}$, we have $< \frac{e^{yt}-1}{t} \mid p(x) >= \int_0^y p(u)du$ $(see[4, 11])$

Recently, Dere and Simsek have studied umbral calculus related to special polynomials. In this paper, we study some properties of associated sequences in umbral algebra. From these properties, we derive new and interesting identities of several kinds of polynomials.

2. Some identities of polynomials arising from umbral calculus

Let $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$. Then, for $n \geq 1$, we have

$$q_n(x) = x \left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x), \quad (see[11]). \tag{1}$$

Let us take $p_n(x) = (x)_n$ and $q_n(x) = x^n$. Then we see that $(x)_n \sim (1, e^t - 1)$ and $x^n \sim (1, t)$.

It is easy to show that

$$\left(\frac{e^t-1}{t}\right)^n = \underbrace{\left(\frac{e^t-1}{t}\right) \dots \left(\frac{e^t-1}{t}\right)}_{n-times} = \sum_{l=0}^{\infty} \frac{n!}{(l+n)!} S_2(l+n, n) t^l, \tag{2}$$

where $S_2(n, k)$ is the stirling number of the second kind.

For $n \geq 1$, by (1), we get

$$\begin{aligned} x^n &= x \left(\frac{e^t-1}{t}\right)^n x^{-1}(x)_n \\ &= x \left(\frac{e^t-1}{t}\right)^n (x-1)_{n-1} \\ &= x \sum_{l=0}^{\infty} \frac{n!}{(l+n)!} S_2(l+n, n) t^l (x-1)_{n-1}, \end{aligned} \tag{3}$$

where $(x)_n = x(x-1)...(x-n+1)$.
The stirling number of the first kind is defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad (see[6,11]). \tag{4}$$

By (3) and (4), we get

(5)

$$\begin{aligned}(x+1)^n &= \sum_{l=0}^{n} \frac{(n+1)!}{(l+n+1)!} S_2(l+n+1,n+1) \sum_{m=0}^{n-l} S_1(n,l+m)x^m (l+m)_l \\ &= \sum_{m=0}^{n} \sum_{l=0}^{n-m} \frac{\binom{l+m}{l}}{\binom{l+n+1}{l}} S_2(l+n+1,n+1)S_1(n,l+m)x^m,\end{aligned}$$

and

$$(x+1)^m = \sum_{m=0}^{n} \binom{n}{m} x^m. \tag{6}$$

Therefore , by (5) and (6), we obtain the following theorem.

Theorem 1 . For $m,n \in \mathbb{Z}_+ = \mathbb{N} \bigcup \{0\}$ with $n \geq m \geq 0$, we have

$$\binom{n}{m} = \sum_{l=0}^{n-m} \frac{\binom{l+m}{l}}{\binom{l+n+1}{l}} S_2(l+n+1,n+1)S_1(n,l+m).$$

If is known that

$$x^n \sim (1,t), \quad (x)_n = (1, e^t - 1), \quad (see[11]). \tag{7}$$

By (1) and (7), we get

$$\begin{aligned}(x)^n &= x\left(\frac{t}{e^t-1}\right)^n x^{-1}x^n \\ &= x\left(\frac{t}{e^t-1}\right)^n x^{n-1} = xB_{n-1}^{(n)}(x).\end{aligned} \tag{8}$$

Thus, by (8), we have

$$B_{n-1}^{(n)}(x) = (x-1)_{n-1}, (n \in \mathbb{N}). \tag{9}$$

Therefore, by(9), we obtain the following lemma.

Lemma 2 . For $n \in \mathbb{N}$, we have

$$B_n^{(n+1)}(x+1) = (x)_n.$$

Note that

$$\sum_{l=0}^{\infty}\left(\frac{e^t-1}{t}\right)B_l^{(n+1)}(x)\frac{t^l}{l!}=\left(\frac{e^t-1}{t}\right)\left(\frac{t}{e^t-1}\right)^{n+1}e^{xt} \tag{10}$$
$$=\left(\frac{t}{e^t-1}\right)^{n}e^{xt}$$
$$=\sum_{l=0}^{\infty}B_l^{(n)}(x)\frac{t^l}{l!}.$$

By comparing the coefficients on the both sides of (10), we get

$$\left(\frac{e^t-1}{t}\right)B_l^{(n+1)}(x)=B_l^{(n)}(x),\quad (l\geq 0). \tag{11}$$

From (11), we have

$$\left(\frac{e^t-1}{t}\right)B_n^{(n+1)}(x)=B_n^{(n)}(x),\quad (n\geq 0). \tag{12}$$

By Lemma 2, (11) and (12), we get

$$B_n^{(n)}(x+1)=\left(\frac{e^t-1}{t}\right)B_n^{(n+1)}(x+1) \tag{13}$$
$$=\left(\frac{e^t-1}{t}\right)(x)_n$$
$$=\int_x^{x+1}(u)_n du.$$

From (4) and (13), we have

$$\int_x^{x+1}(u)_n du=\sum_{l=0}^{n}S_1(n,l)\int_{x-1}^{x}u^l du \tag{14}$$
$$=\sum_{l=0}^{n}\frac{S_1(n,l)}{l+1}(x^{l+1}-(x-1)^{l+1}).$$

Therefore, by (13) and (14), we obtain the following theorem.

Theorem 3 . For $n\geq 1$, we have

$$B_n^{(n)}(x+1)=\sum_{l=0}^{n}S_1(n,l)\frac{1}{l+1}(x^{l+1}-(x-1)^{l+1}).$$

For $a\neq 0$, Abel sequence is defined by $A_n(x;a)=x(x-an)^{n-1}$. In [11], we note that $A_n(x;a)\sim(1,te^{at})$.

Let us consider the following associated sequences:

$$A_n(x;a) = x(x-an)^{n-1} \sim (1, te^{at}), \quad a \neq 0, \tag{15}$$
$$(\frac{x}{b})_n \sim (1, e^{bt}-1), \quad (b \neq 0).$$

For $n \geq 1$, by(11), we get

$$\begin{aligned}(\frac{x}{b})_n &= x\left(\frac{te^{at}}{e^{bt}-1}\right)^n x^{-1}A_n(x;a) \\ &= \frac{x}{b^n}\left(\frac{bt}{e^{bt}-1}\right)^n e^{ant}(x-an)^{n-1},\end{aligned} \tag{16}$$

where

$$\left(\frac{bt}{e^{bt}-1}\right)^n e^{ant} = \sum_{k=0}^{\infty} b^k B_k^{(n)}\left(\frac{an}{b}\right)\frac{t^k}{k!}. \tag{17}$$

From (16) and (17), we have

$$\begin{aligned}(\frac{x}{b})_n &= \frac{x}{b^n}\left(\sum_{k=0}^{\infty} b^k B_k^{(n)}\left(\frac{an}{b}\right)\frac{t^k}{k!}\right)(x-an)^{n-1} \\ &= x\sum_{k=0}^{n-1} b^{k-n}B_k^{(n)}\left(\frac{an}{b}\right)\frac{(n-1)_k}{k!}(x-an)^{n-1-k} \\ &= \sum_{k=0}^{n-1} b^{k-n}B_k^{(n)}\left(\frac{an}{b}\right)\binom{n-1}{k}x(x-an)^{n-1-k} \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}b^{k-n}B_k^{(n)}\left(\frac{an}{b}\right)A_{n-k}(x-ak;a).\end{aligned} \tag{18}$$

On the other hand, by (16), we get

$$\begin{aligned}(\frac{x}{b})_n &= \left(\frac{1}{b}\right)^n x\left(\frac{bt}{e^{bt}-1}\right)^n e^{ant}(x-an)^{n-1} \\ &= \left(\frac{1}{b}\right)^n x\left(\frac{bt}{e^{bt}-1}\right)^n x^{n-1} \\ &= \left(\frac{1}{b}\right)^n x\sum_{l=0}^{n-1}\frac{B_l^{(n)}}{l!}b^l(n-1)_l x^{n-l-1} \\ &= \frac{x}{b}\sum_{l=0}^{n-1}\binom{n-1}{l}\left(\frac{1}{b}\right)^{n-l-1}x^{n-l-1}B_l^{(n)} \\ &= \frac{x}{b}B_{n-1}^{(n)}(\frac{x}{b}).\end{aligned} \tag{19}$$

Therefore, by (18) and (19), we obtain the following theorem.

Theorem 4 . For $n \geq 1$, we have

$$\begin{aligned}(\frac{x}{b})_n &= \sum_{k=0}^{n-1} \binom{n-1}{k} b^{k-n} B_k^{(n)} \left(\frac{an}{b}\right) A_{n-k}(x-ak;a) \\ &= \frac{x}{b} B_{n-1}^{(n)}(\frac{x}{b}).\end{aligned}$$

Moreover,

$$xB_{n-1}^{(n)}(\frac{x}{b}) = \sum_{k=0}^{n-1} \binom{n-1}{k} b^{k-n+1} B_k^{(n)} \left(\frac{an}{b}\right) A_{n-k}(x-ak;a).$$

Remark . For $b=1$, $n \geq 1$, we have

$$(x)_n = xB_{n-1}^{(n)}(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k^{(n)}(an) A_{n-k}(x-ak;a).$$

Let $\phi_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k$ be exponential polynomial.
Then, we note that

$$\phi_n(x) \sim (1, \log(1+t)), \quad x^n \sim (1,t). \tag{20}$$

It is well known that

$$\left(\frac{\log(1+t)}{t}\right)^n = n \sum_{k=0}^{\infty} \frac{B_k^{(n+k)}}{n+k} \frac{t^k}{k!}, (see[5,6]). \tag{21}$$

By (1) and (20), we get

$$\begin{aligned} x^n &= x \left(\frac{\log(1+t)}{t}\right)^n x^{-1}\phi_n(x) \\ &= x\{n \sum_{k=0}^{\infty} \frac{B_k^{(n+k)}}{k+n} \frac{t^k}{k!}\} x^{-1}\phi_n(x) \\ &= n \sum_{k=0}^{n-1} \sum_{l=k+1}^{n} \frac{\binom{l-1}{k}}{n+k} B_k^{(n+k)} S_2(n,l) x^{l-k} \\ &= n \sum_{k=0}^{n-1} \sum_{m=1}^{n-k} \frac{\binom{k+m-1}{k}}{n+k} B_k^{(n+k)} S_2(n,k+m) x^m \\ &= n \sum_{m=1}^{n} \{ \sum_{k=0}^{n-m} \frac{\binom{k+m-1}{k}}{n+k} B_k^{(n+k)} S_2(n,k+m)\} x^m. \end{aligned} \tag{22}$$

Thus, by (22), we obtain the following theorem.

Theorem 5 . For $n \geq 1$ with $1 \leq m \leq n$, we have

$$\sum_{k=0}^{n-m} \frac{n\binom{k+m-1}{k}}{n+k} B_k^{(n+k)} S_2(n,k+m) = \delta_{m,n}.$$

Let $M_n(x) = \sum_{k=0}^{n} \binom{n}{k}(n-1)_{n-k}2^k(x)_k$ be Mittag-Leffler polynomials with $M_n(x) \sim (1, \frac{e^t-1}{e^t+1})$. Then, let us consider the following associated sequence:

$$M_n(x) \sim (1, \frac{e^t-1}{e^t+1}), \quad (x)_n = (1, e^t-1). \tag{23}$$

For $n \geq 1$, by (1) and (23), we get

$$\begin{aligned}(x)_n &= x\left(\frac{1}{e^t+1}\right)^n x^{-1}M_n(x) \\ &= \sum_{k=0}^{n}\binom{n}{k}(n-1)_{n-k}2^k x\left(\frac{1}{e^t+1}\right)^n (x-1)_{k-1} \\ &= \sum_{k=0}^{n}\sum_{l=0}^{k-1}\binom{n}{k}(n-1)_{n-k}2^k S_1(k-1,l)\frac{1}{2^n}x\left(\frac{2}{e^t+1}\right)^n (x-1)^l \\ &= \sum_{k=0}^{n}\sum_{l=0}^{k-1}\binom{n}{k}(n-1)_{n-k}2^{k-n}S_1(k-1,l)xE_l^{(n)}(x-1).\end{aligned} \tag{24}$$

Thus, by (24), we obtain the following proposition.

Proposition 6 . For $n \geq 1$, we have

$$(x)_n = \sum_{k=0}^{n}\sum_{l=0}^{k-1}\binom{n}{k}(n-1)_{n-k}2^{k-n}S_1(k-1,l)xE_l^{(n)}(x-1).$$

For $n \geq 1$, by (1) and (23), we get

$$\begin{aligned}M_n(x) &= x(e^t+1)^n x^{-1}(x)_n = x(e^t+1)^n(x-1)_{n-1} \\ &= x\sum_{k=0}^{n}\binom{n}{k}e^{kt}(x-1)_{n-1} = x\sum_{k=0}^{n}\binom{n}{k}(x+k-1)_{n-1}.\end{aligned} \tag{25}$$

The equation (25) is different from the expression

$$M_n(x) = \sum_{k=0}^{n}\binom{n}{k}(n-1)_{n-k}2^k(x)_k.$$

Therefore, by (25), we obtain the following corollary.

Corollary 7 . For $n \geq 1$, we have

$$M_n(x) = x\sum_{k=0}^{n}\binom{n}{k}(x+k-1)_{n-1}.$$

Let $L_n^{(\alpha)}(x)$ be the Laguerre polynomials of order $\alpha(\in \mathbb{R})$. Then we note that $L_n^{(\alpha)}(x) \sim ((1-t)^{-\alpha-1}, \frac{t}{t-1})$. Especially, $L_n(x) \sim (1, \frac{t}{t-1})$. By the

definition of associated sequences, we see that

$$< \left(\frac{t}{t-1}\right)^n | \quad L_n(x) >= n!\delta_{n,k}, \quad (n,k \geq 0). \tag{26}$$

From (26), we have

$$< \left(\frac{t}{t+1}\right)^n | \quad L_n(-x) >= n!\delta_{n,k}. \tag{27}$$

Thus, by (27), we get

$$L_n(-x) \sim (1, \frac{t}{t+1}). \tag{28}$$

As it is shown in Roman [11], one can find an explicit expression for $L_n(x)$ by using the transfer formula

$$L_n(-x) = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{n!}{k!} x^k, \quad (n \geq 1), (see[11]). \tag{29}$$

It is well known that

$$\frac{t}{(1+t)\log(1+t)} = \sum_{k=0}^{\infty} B_k^{(k)} \frac{t^k}{k!}, (see[5,6]). \tag{30}$$

Thus, by (29), we get

$$\left(\frac{t}{(1+t)\log(1+t)}\right)^n = \sum_{k=0}^{\infty} \left(\sum_{l_1+\ldots+l_n=k} \binom{k}{l_1,\ldots,l_n} B_{l_1}^{(l_1)} \ldots B_{l_n}^{(l_n)} \right) \frac{t^k}{k!}. \tag{31}$$

From (1),(20) and (28), we have

$$\begin{aligned} \phi_n(x) &= x \left(\frac{t}{(1+t)\log(1+t)}\right)^n x^{-1} L_n(-x) \\ &= x \left(\frac{t}{(1+t)\log(1+t)}\right)^n x^{-1} \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{n!}{m!} x^m. \end{aligned} \tag{32}$$

By (30) and (31), we get

(33)

$$\begin{aligned} &\phi_n(x) \\ &= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{n!}{m!} x \{ \sum_{k=0}^{m-1} \sum_{l_1+\ldots+l_n=k} \binom{k}{l_1,\ldots,l_n} B_{l_1}^{(l_1)} \ldots B_{l_n}^{(l_n)} \} \frac{t^k}{k!} x^{m-1} \\ &= \sum_{m=1}^{n} \sum_{k=0}^{m-1} \sum_{l_1+\ldots+l_n=k} \binom{n-1}{m-1} \binom{m-1}{k} \frac{n!}{m!} \binom{k}{l_1,\ldots,l_n} B_{l_1}^{(l_1)} \ldots B_{l_n}^{(l_n)} x^{m-k} \\ &= \sum_{m=1}^{n} \{ \sum_{l=1}^{m} \sum_{l_1+\ldots+l_n=m-l} \binom{n-1}{m-1} \binom{m-1}{l-1} \frac{n!}{m!} \binom{m-l}{l_1,\ldots,l_n} B_{l_1}^{(l_1)} \ldots B_{l_n}^{(l_n)} \} x^l. \end{aligned}$$

From (20), we have

$$\phi_n(x)=\sum_{k=0}^{n}S_2(n,k)x^k=\sum_{k=1}^{n}S_2(n,k)x^k,\quad (n\geq 1). \tag{34}$$

Therefore, by (32) and (33), we obtain the following theorem.

Theorem 8 . For $n\geq 1$ with $1\leq l\leq n$, we have

$$S_2(n,l)=\sum_{l\leq m\leq n}\sum_{l_1+\ldots+l_n=m-l}\binom{n-1}{m-1}\binom{m-1}{l-1}\frac{n!}{m!}\binom{m-l}{l_1,\ldots,l_n}B_{l_1}^{(l_1)}\ldots B_{l_n}^{(l_n)}.$$

It is well known in [5,6] that

$$\frac{(e^t-1)^n}{e^{tx}t^n}=(n!)^2\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}(-1)^{k-j}\frac{\binom{k}{j}S_2(j+n,n)}{\binom{j+n}{j}}x^{k-j}\right)\frac{t^k}{k!}. \tag{35}$$

From (35), we have

(36)

$$\left(\frac{e^{bt}-1}{te^{at}}\right)^n=(n!)^2b^n\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}(-1)^{k-j}\frac{\binom{k}{j}S_2(j+n,n)}{\binom{j+n}{j}}(an)^{k-j}b^j\right)\frac{t^k}{k!},$$

where $a,b\neq 0$.
By (1) and (15), we get

(37)

$$\begin{aligned}A_n(x;a)&=x\left(\frac{e^{bt}-1}{te^{at}}\right)^n x^{-1}(\frac{x}{b})_n\\&=x\left(\frac{e^{bt}-1}{te^{at}}\right)^n\frac{1}{b}(\frac{x}{b}-1)_{n-1}\\&=(n!)^2b^{n-1}x\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}(-an)^{k-j}b^j\frac{\binom{k}{j}S_2(j+n,n)}{\binom{j+n}{j}}\right)\frac{t^k}{k!}(\frac{x}{b}-1)_{n-1},\end{aligned}$$

where

$$\begin{aligned}t^k(\frac{x}{b}-1)_{n-1}&=\sum_{l=0}^{n-1}S_1(n-1,l)t^k(\frac{x}{b}-1)^l\\&=\sum_{l=k}^{n-1}S_1(n-1,l)(\frac{1}{b})^k(l)_k(\frac{x}{b}-1)^{l-k}.\end{aligned} \tag{38}$$

From (36) and (37), we have

(39)

$$A_n(x;a) = x(x-an)^{n-1}$$
$$= (n!)^2 b^{n-1} \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} \sum_{l=k}^{n-1} \left(-\frac{an}{b}\right)^{k-j} \frac{\binom{k}{j}\binom{l}{k} S_2(j+n,n) S_1(n-1,l)}{\binom{j+n}{j}} \right) x(\frac{x}{b}-1)^{l-k}.$$

Therefore, by (39), we obtain the following lemma.

Lemma 9 . For $n \geq 1$, we have

$$A_n(x;a)$$
$$= (n!)^2 b^{n-1} \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} \sum_{l=k}^{n-1} \left(-\frac{an}{b}\right)^{k-j} \frac{\binom{k}{j}\binom{l}{k} S_2(j+n,n) S_1(n-1,l)}{\binom{j+n}{j}} \right) x(\frac{x}{b}-1)^{l-k}.$$

Remark. Let $b = 1$. Then we have

$$A_n(x;a) = x(x-an)^{n-1}$$
$$= (n!)^2 \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} \sum_{l=k}^{n-1} (-an)^{k-j} \frac{\binom{k}{j}\binom{l}{k} S_2(j+n,n) S_1(n-1,l)}{\binom{j+n}{j}} \right) x(x-1)^{l-k}.$$

It is well known in[5,6] that

$$\frac{(1+t)^{x-1}t^n}{(\log(1+t))^n} = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(x)\frac{t^k}{k!}. \tag{40}$$

From (39), we have

$$\left(\frac{t(1+t)^a}{\log(1+t)}\right)^n = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(an+1)\frac{t^k}{k!}. \tag{41}$$

Let us consider the following associated sequences:

$$S_n(x) \sim (1, t(t+1)^a), (a \neq 0), \quad x^n \sim (1,t). \tag{42}$$

Then, for $n \geq 1$, by (1) and (41), we get

$$\begin{aligned} S_n(x) &= x\left(\frac{t}{t(1+t)^a}\right)^n x^{-1}x^n \\ &= x(1+t)^{-an}x^{n-1} \\ &= x\sum_{k=0}^{n-1}\binom{-an}{k}(n-1)_k x^{n-1-k} \\ &= \sum_{k=1}^{n}\binom{-an}{n-k}(n-1)_{n-k}x^k. \end{aligned} \tag{43}$$

Therefore, by (42), we obtain the following proposition.

Proposition 10 . For $n \geq 1$, let $S_n(x) \sim (1, t(1+t)^a)$, $(a \neq 0)$. Then, we have

$$S_n(x) = \sum_{k=1}^{n} \binom{-an}{n-k} (n-1)_{n-k} x^k.$$

By (1), (20) and (41), we get

$$\begin{aligned} \phi_n(x) &= x \left(\frac{t(1+t)^a}{\log(1+t)} \right)^n x^{-1} S_n(x) \\ &= x \left(\frac{t(1+t)^a}{\log(1+t)} \right)^n \sum_{l=1}^{n} \binom{-an}{n-l} (n-1)_{n-l} x^{l-1}. \end{aligned} \tag{44}$$

From (40) and (44), we get

(45)

$$\begin{aligned} \phi_n(x) &= \sum_{l=1}^{n} \binom{-an}{n-l} (n-1)_{n-l} x \left(\frac{t(1+t)^a}{\log(1+t)} \right)^n x^{l-1} \\ &= \sum_{l=1}^{n} \binom{-an}{n-l} (n-1)_{n-l} \sum_{k=0}^{l-1} B_k^{(k-n+1)}(an+1) \frac{1}{k!} x t^k x^{l-1} \\ &= \sum_{l=1}^{n} \sum_{k=0}^{l-1} \binom{-an}{n-l} (n-1)_{n-l} \binom{l-1}{k} B_k^{(k-n+1)}(an+1) x^{l-k} \\ &= \sum_{l=1}^{n} \sum_{m=1}^{l} \binom{-an}{n-l} (n-1)_{n-l} \binom{l-1}{m-1} B_{l-m}^{(l-n-m+1)}(an+1) x^m \\ &= \sum_{m=1}^{n} \{ \sum_{l=m}^{n} \binom{-an}{n-l} (n-1)_{n-l} \binom{l-1}{m-1} B_{l-m}^{(l-n-m+1)}(an+1) \} x^m. \end{aligned}$$

Therefore, by (20) and (44), we obtain the following theorem.

Theorem 11 . For $n \geq 1$ and $m \geq 0$, we have

$$S_2(n,m) = \sum_{m \leq l \leq n} \binom{-an}{n-l} (n-1)_{n-l} \binom{l-1}{m-1} B_{l-m}^{(l-m-n+1)}(an+1).$$

Let us consider the following associated sequences:

$$S_n^*(x) \sim (1, \frac{t(e^t+1)}{2}), \quad x^n \sim (1,t) \tag{46}$$

Then, by (1) and (46), we get

$$S_n^*(x) = x \left(\frac{2}{e^t+1} \right)^n x^{n-1} = x E_{n-1}^{(n)}(x). \tag{47}$$

For $n \geq 1$, by (1), (15) and (46), we get

$$xE_{n-1}^{(n)}(x) = x\left(\frac{2}{e^t+1}\right)^n e^{ant}(x-an)^{n-1} \tag{48}$$
$$= x\sum_{k=0}^{n-1}\frac{E_k^{(n)}(an)}{k!}t^k(x-an)^{n-1}$$
$$= \sum_{k=0}^{n-1}\binom{n-1}{k}E_k^{(n)}(an)x(x-an)^{n-1-k}.$$

Therefore, by (47) , we obtain the following theorem.

Theorem 12 . For $n \geq 1$, we have

$$xE_{n-1}^{(n)}(x) = \sum_{k=0}^{n}\binom{n-1}{k}E_k^{(n)}(an)A_{n-k}(x-ak;a).$$

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